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Derivation of Yang – Mills equations from Maxwell Equations and Exact solutions

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Abstract

In this paper we derived the Yang-Mills equations from Maxwell equations. Consequently we find a new form for self-duality equations. In addition exact solution class of the classical SU(2) Yang-Mills field equations in four-dimensional Euclidean space and two exact solution classes for SU(2) Yang- Mills equations when is ρ a complex analytic function are also obtained.

Keywords : Self-dual SU(2); Yang-Mills fields; Gauge theory

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1. Introduction

The self-dual Yang-Mills equations (a system of equations for Lie algebra valued functions of C^4) play a central role in the field of integrable systems and also play a fundamental role in several other areas of mathematics and physics [1-4].

In addition the self-dual Yang-Mills equations are of great importance in their own right and have found a remarkable number of applications in physics and mathematics as well. These equations arise in the context of gauge theory [5], in classical general relativity [6,7], and can be used as a powerful tool in the analysis of 4-manifolds [8].

Non-Abelian gauge theories first appeared in the seminal work of Yang and Mills (1954) as a non-Abelian generalization of Maxwells equations [9]. The fact that the Yang-Mills equations have a natural geometric interpretation was recognized early on in the history of gauge theory [10,11].

The Yang-Mills equations are a set of coupled, second-order partial differential equations in four dimensions for the Lie algebra-valued gauge potential functions A_μ, and are extremely difficult to solve in general. The self-dual Yang-Mills equations describe a connection for a bundle over the Grassmannian of two-dimensional subspaces of the twister space [12.13].

A very important property of the theory of non-abelian gauge fields is that the action functional has local minima in the Euclidean domain with non-vanishing field strength $F_{\mu\nu}$ [14]. The corresponding field configurations, which are often called pseudoparticles, have the self-dual or anti-self-dual field strength, and fall into topologically inequivalent classes labelled by an integer n, the Pontryagin index. The existence of these non-local minima was first pointed out by Belavin et al.(1975) who also exhibited the solution of the self-duality equation with $n = 1$ for an $SU(2)$ gauge group [15]. Solutions of the self-duality equations with an arbitrary number of pesudoparticles were discovered by Witten (1979) and t' Hooft (1979) [16].

In this paper we found a new representation for self-duality equations. In addition exact solution class of the classical $SU(2)$ Yang-Mills field equations in four-dimensional Euclidean space and two exact solution classes for $SU(2)$ Yang-Mills equations when ρ is a complex analytic function are also obtained.

This paper is organized as follows: This introduction followed by the derivation of Yang – Mills equations from Maxwell Equations in section 2. A new representation of the self-duality equations in section 3. In section 4 we found an exact solution class of the classical $SU(2)$ Yang-Mills field equations. Moreover two exact solution classes for self-dual $SU(2)$ gauge fields on Euclidean space when ρ is a complex analytic function are given in section 5.

Derivation of Yang –Mills equations from Maxwell Equations

The classical equations of Maxwell describing electromagnetic phenomena are

$$\nabla . E = 4\pi\rho\,, \quad \nabla \times B = 4\pi J + \frac{\partial E}{\partial t}\,,$$

$$\nabla . B = 0\,, \quad \nabla \times E = -\frac{\partial B}{\partial t}\,,$$

We would like to formulate these equations in the language of differential forms. Let $x_\mu = (t, x^1, x^2, x^3)$ be local coordinates in Minkowski's space $M_{1,3}$. Define the Maxwell 2-form F by the equation

$$F = \frac{1}{2} F_{\mu v} dx^\mu dx^v, \qquad (\mu, v = 0,1,2,3), \tag{2}$$

where

$$F_{\mu v} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{3}$$

Written in complete detail, Maxwell's 2-form is given by

$$F = -E_x dt \wedge dx^1 - E_y dt \wedge dx^2 - E_z dt \wedge dx^3 +$$
$$B_z dx^1 \wedge dx^2 - B_y dx^1 \wedge dx^3 + B_x dx^2 \wedge dx^3\ . \tag{4}$$

We also define the source current 1-form

$$J = J_\mu dx^\mu = \rho dt + J_1 dx^1 + J_2 dx^2 + J_3 dx^3. \tag{5}$$

Proposition 1: Maxwell's Equations (1) are equivalent to the equations

$$dF = 0,$$

$$d * F = 4\pi * J. \tag{6}$$

Proof: The proof is by direct computation using the definitions of the exterior derivative and the Hodge-$*$ operator.

$$dF = -\frac{\partial E_x}{\partial x^2} \wedge dx^2 \wedge dt \wedge dx^1 - \frac{\partial E_x}{\partial x^3} \wedge dx^3 \wedge dt \wedge dx^1 - \frac{\partial E_y}{\partial x^1} \wedge dx^1 \wedge dt \wedge dx^2 +$$
$$-\frac{\partial E_y}{\partial x^3} \wedge dx^3 \wedge dt \wedge dx^2 - \frac{\partial E_z}{\partial x^1} \wedge dx^1 \wedge dt \wedge dx^3 - \frac{\partial E_z}{\partial x^2} \wedge dx^2 \wedge dt \wedge dx^3 +$$
$$\frac{\partial B_z}{\partial t} \wedge dt \wedge dx^1 \wedge dx^2 - \frac{\partial B_z}{\partial x^3} \wedge dx^3 \wedge dx^1 \wedge dx^2 - \frac{\partial B_y}{\partial t} \wedge dt \wedge dx^1 \wedge dx^3 +$$
$$-\frac{\partial B_y}{\partial x^2} \wedge dx^2 \wedge dx^1 \wedge dx^3 + \frac{\partial B_x}{\partial t} \wedge dt \wedge dx^2 \wedge dx^3 + \frac{\partial B_x}{\partial x^1} \wedge dx^1 \wedge dx^2 \wedge dx^3.$$

Collecting terms and using the anti-symmetry of the wedge operator, we get

$$dF = \left(\frac{\partial B_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_z}{\partial x^3}\right) dx^1 \wedge dx^2 \wedge dx^3 + \left(\frac{\partial E_y}{\partial x^3} - \frac{\partial E_z}{\partial x^2} - \frac{\partial B_x}{\partial t}\right) dx^2 \wedge dt \wedge dx^3 +$$
$$\left(\frac{\partial E_z}{\partial x^1} - \frac{\partial E_x}{\partial x^3} - \frac{\partial B_y}{\partial t}\right) dt \wedge dx^1 \wedge dx^3 + \left(\frac{\partial E_x}{\partial x^2} - \frac{\partial E_y}{\partial x^1} - \frac{\partial B_z}{\partial t}\right) dx^1 \wedge dt \wedge dx^2.$$

Therefore, $\mathrm{dF} = 0$ iff

$$\frac{\partial B_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_z}{\partial x^3} = 0,$$

which is the same as

$$\nabla . B = 0\,,$$

and

$$\frac{\partial E_y}{\partial x^3} - \frac{\partial E_z}{\partial x^2} - \frac{\partial B_x}{\partial t} = 0,$$
$$\frac{\partial E_z}{\partial x^1} - \frac{\partial E_x}{\partial x^3} - \frac{\partial B_y}{\partial t} = 0,$$
$$\frac{\partial E_x}{\partial x^2} - \frac{\partial E_y}{\partial x^1} - \frac{\partial B_z}{\partial t} = 0,$$

which means that

$$-\nabla \times E - \frac{\partial B}{\partial t} = 0\,, \tag{7}$$

To verify the second set of Maxwell equations, we first compute the dual of the current density 1-form (5) using the results of

$$* dt = -dx^1 \wedge dx^2 \wedge dx^3$$
$$* dx^1 = dx^2 \wedge dt \wedge dx^3$$
$$* dx^2 = dt \wedge dx^1 \wedge dx^3$$
$$* dx^3 = dx^1 \wedge dt \wedge dx^2.$$

We get

$$* J = -\rho dx^1 \wedge dx^2 \wedge dx^3 + J_1 dx^2 \wedge dt \wedge dx^3 + J_2 dt \wedge dx^1 \wedge dx^3 + J_3 dx^1 \wedge dt \wedge dx^2. \tag{8}$$

We could now proceed to compute $d * F$, but perhaps it is more elegant to notice that $F \in \Lambda^2(M)$, and F splits into $F = F+ + F-$. In fact, we see from (3) that the components of $F+$ are those of $-E$ and the components of $F-$ constitute the magnetic field vector B. Using the above results, we can immediately write the components of $* F$:

$$F = B_x dt \wedge dx^1 + B_y dt \wedge dx^2 + B_z dt \wedge dx^3 + E_z dx^1 \wedge dx^2 - E_y dx^1 \wedge dx^3 + E_x dx^2 \wedge dx^3, \quad (9)$$

or equivalently,

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{bmatrix} . \quad (10)$$

Since the effect of the dual operator amounts to exchanging

$$E \longmapsto -B$$

$$B \longmapsto +E,$$

we can infer from equations (7) and (8) that

$$\nabla . E = 4\pi\rho ,$$

and

$$\nabla \times B - \frac{\partial E}{\partial t} = 4\pi J .$$

2. New form of the self-duality equations

The essential idea of Yang and Mills (1954) is to consider an analytic continuation of the gauge potential A_μ into complex space where x_1, x_2, x_3 and x_4 are complex. The self-duality equations $F_{\mu\nu}={}^*F_{\mu\nu}$ are then valid also in complex space , in a region containing real space where the x' s are real. Now consider four new complex variables , $\bar{\mathcal{Y}}$, $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ defined by

$$\sqrt{2}\mathcal{Y} = x_1 + ix_2 , \qquad \sqrt{2}\bar{\mathcal{Y}} = x_1 - ix_2 ,$$
$$\sqrt{2}\mathcal{Z} = x_3 - ix_4 , \qquad \sqrt{2}\bar{\mathcal{Z}} = x_3 + ix_4 , \quad (11)$$

it is simple to check that the self-duality equations $F_{\mu\nu} = {}^*F_{\mu\nu}$ reduces to

$$F_{\mathcal{Y}\mathcal{Z}} = 0, \qquad F_{\bar{\mathcal{Y}}\bar{\mathcal{Z}}} = 0 , \qquad F_{\mathcal{Y}\bar{\mathcal{Y}}} + F_{\mathcal{Z}\bar{\mathcal{Z}}} = 0. \quad (12)$$

Equations (12) can be immediately integrated, since they are pure gauge, to give [17,18,19]

$$A_{\mathcal{Y}} = D^{-1}D_{\mathcal{Y}} , \qquad A_{\mathcal{Z}} = D^{-1}D_{\mathcal{Z}} , \qquad A_{\bar{\mathcal{Y}}} = \bar{D}^{-1}\bar{D}_{\bar{\mathcal{Y}}} , \qquad A_{\bar{\mathcal{Z}}} = \bar{D}^{-1}\,\bar{D}_{\bar{\mathcal{Z}}} , \quad (13)$$

where D and $\bar{D}$ are arbitrary 2 × 2 complex matrix functions of $\mathcal{Y}$, $\bar{\mathcal{Y}}$, $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ with determinant = 1 (for $SU(2)$ gauge group) and $D_{\mathcal{Y}} = \partial_{\mathcal{Y}} D$, etc. For real gauge fields $A_\mu \doteq -A_\mu^+$ (the symbol $\doteq$ is used for equations valid only for real values of x_1 , x_2 , x_3 and x_4), we rquire

$$\bar{D} \doteq (D^+)^{-1}. \quad (14)$$

Gauge transformations are the transformations

$$D \longrightarrow , \qquad \bar{D} \rightarrow \bar{D}U , \qquad U^+U \doteq I, \quad (15)$$

where U is a 2 × 2 matrix function of $\mathcal{Y}$, $\bar{\mathcal{Y}}$, $\mathcal{Z}$, $\bar{\mathcal{Z}}$ with determined = 1. Under transformtion (15), equation (14) remains un-changed.. We now define the hermitian matrix $\mathcal{J}$ as [20-22]

$$\mathcal{J} \equiv D\,\bar{D}^{-1} \doteq D\,D^+. \quad (16)$$

$\mathcal{J}$ has the very important property of being invariant under the gauge transformation (15). The only nonvanishing field strengths in terms of $\mathcal{J}$ becomes

$$F_{u\bar{v}} = -\bar{D}^{-1}\,(\mathcal{J}^{-1}\,\mathcal{J}_u)_{\bar{v}}\bar{D}, \quad (17)$$

(u, v = $\mathcal{Y}$, $\mathcal{Z}$) and the remaining self-duality equation (12) takes the form:

$$(\mathcal{J}^{-1}\mathcal{J}_{\mathcal{Y}})_{\bar{\mathcal{Y}}} + (\mathcal{J}^{-1}\mathcal{J}_{\mathcal{Z}})_{\bar{\mathcal{Z}}} = 0. \quad (18)$$

The action density in terms of $\mathcal{J}$ [23] is

$$\phi(\mathcal{J}) \equiv -\frac{1}{2}\, Tr\, F_{\mu\nu}F_{\mu\nu} \quad = -2\mathrm{Tr}\,(\; F_{y\bar{y}}F_{z\bar{z}} + F_{y\bar{z}}F_{\bar{y}z}), \tag{19}$$

where

$$F_{\mu\nu} = \partial_\mu\, A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]\,. \tag{20}$$

Our construction begins by explicit parametrization of the matrix $\mathcal{J}$

$$\mathcal{J} = \begin{pmatrix} \phi & \rho \\ \bar{\rho} & \phi(1+\rho\bar{\rho}) \end{pmatrix}, \tag{21}$$

and for real gauge fields $A_\mu \doteq -A_\mu^+$, we require ϕ $\doteq$real, $\bar{\rho} \doteq \rho^*$ ($\rho^* \equiv$ complex conjugate of ρ). The self-duality equations (18) take the form

$$\frac{1}{2}(1+\rho\bar{\rho})\partial_\mu\partial_\mu ln\emptyset + \frac{1}{2}\rho\partial_\mu\partial_\mu\bar{\rho} + \frac{\bar{\rho}}{\emptyset}[\emptyset_y\rho_{\bar{y}} + \emptyset_z\rho_{\bar{z}}] + \frac{\rho}{\emptyset}[\emptyset_y\bar{\rho}_{\bar{y}} + \emptyset_z\bar{\rho}_{\bar{z}}] +$$
$$+[\rho_{\bar{y}}\bar{\rho}_y + \rho_{\bar{z}}\,\bar{\rho}_z] = 0, \tag{22}$$

$$\frac{\phi}{2}\partial_\mu\partial_\mu\bar{\rho} + \frac{\bar{\rho}}{2}\partial_\mu\partial_\mu\emptyset - \frac{2\bar{\rho}}{\phi}\left(\emptyset_y\emptyset_{\bar{y}} + \emptyset_z\phi_{\bar{z}}\right) + \left(\emptyset_y\bar{\rho}_{\bar{y}} + \emptyset_z\bar{\rho}_{\bar{z}} - \emptyset_{\bar{y}}\bar{\rho}_y - \emptyset_{\bar{z}}\bar{\rho}_z\right) = 0\ , \tag{23}$$

$$\frac{\phi}{2}\partial_\mu\partial_\mu\rho + \frac{\rho}{2}\partial_\mu\partial_\mu\emptyset - \frac{2\rho}{\phi}\left(\emptyset_y\emptyset_{\bar{y}} + \emptyset_z\phi_{\bar{z}}\right) + \left(\emptyset_{\bar{y}}\rho_y + \emptyset_{\bar{z}}\rho_z - \emptyset_y\rho_{\bar{y}} - \emptyset_z\rho_{\bar{z}}\right) = 0\ , \tag{24}$$

where $\partial_\mu\partial_\mu = 2\left(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}}\right)$.

The positive definite Hermitian matrix $\mathcal{J}$ = D D^+ can be factored into a product upper and lower (or vice versa) triangular matrices as follows

$$\mathcal{J} = R\,R^+ = R^I\,R^{I+},$$
$$R = \begin{pmatrix} \frac{1}{\sqrt{\phi}} & 0 \\ \bar{\rho}\sqrt{\phi} & \sqrt{\phi} \end{pmatrix}, \qquad R^I = \begin{pmatrix} \sqrt{\phi^I} & \bar{\rho}^I\sqrt{\phi^I} \\ 0 & \frac{1}{\sqrt{\emptyset^I}} \end{pmatrix}\ , \tag{25}$$
$$\phi \doteq real, \qquad \bar{\rho} \doteq \rho^*, \qquad \bar{\rho}^I \doteq \rho^{I*}.$$

It is evident from (25) that one can choose a gauge so that D = R or D = R^I and it is easy to check that in both gauges the self-duality equations (22)-(24) (in the case of D = R^I all the ϕ , ρ , $\bar{\rho}$ are replaced by ϕ^I , ρ^I , $\bar{\rho}^I$).

From equation (2.25) we see that $R^{-1}R^I$ is a unitary matrix so that we can always make a gauge transformation from the R gauge to the R^I gauge.

Theorem 1: If ($\emptyset$, ρ, $\bar{\rho}$) satisfy equations (22)-(24) then so do (ϕ^I, ρ^I, $\bar{\rho}^I$) defined by [24]

$$\phi^I = \frac{\phi}{1+\rho\bar{\rho}}\ , \qquad \rho^I = \frac{\bar{\rho}}{1+\rho\bar{\rho}}\ , \qquad \bar{\rho}^I = \frac{\rho}{1+\rho\bar{\rho}}\ ,$$

3. Exact solution class of the classical $SU(2)$Yang-Mills field equations

To obtain an exact solution class of the classical SU(2) Yang-Mills field equations in four-dimensional Euclidean space, consider the system.

$$\frac{1}{2}(1+\rho\bar{\rho})\partial_\mu\partial_\mu ln\phi + \frac{1}{2}\bar{\rho}\partial_\mu\partial_\mu\rho + \frac{\rho}{\phi}[\phi_{\bar{y}}\bar{\rho}_y + \emptyset_{\bar{z}}\bar{\rho}_z] + \frac{\bar{\rho}}{\phi}[\phi_{\bar{y}}\rho_y + \phi_{\bar{z}}\rho_z] +$$
$$+[\bar{\rho}_y\rho_{\bar{y}} + \bar{\rho}_z\rho_{\bar{z}}] = 0, \tag{26}$$
$$\frac{\phi}{2}\partial_\mu\partial_\mu\rho + \frac{\rho}{2}\partial_\mu\partial_\mu\phi - \frac{2\rho}{\phi}\left(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}\right) + \left(\phi_{\bar{y}}\rho_y + \phi_{\bar{z}}\rho_z - \phi_y\rho_{\bar{y}} - \phi_z\rho_{\bar{z}}\right) = 0\ .$$

Let us make the ansatz [25]

$$\phi = \phi(g), \qquad \rho = e^{ia}\sigma(g). \tag{27}$$

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Where $g = g(x_1, x_2, x_3, x_4)$ is a real function of x_μ, $\mu = 1,2,3,4$, ϕ and σ are real functions of g, and a is a real constant. Then equation (26) give the relations

$$(g_{y\bar{y}} + g_{z\bar{z}})\left((1+\sigma^2)\phi^2\right)' - 2(g_y g_{\bar{y}} + g_z g_{\bar{z}})\phi^2\left[(1+\sigma^2)\frac{\phi'}{\phi} + \sigma\sigma'\right]' = 0\,, \tag{28}$$

$$(g_{y\bar{y}} + g_{z\bar{z}})(\phi\sigma)' + (g_y g_{\bar{y}} + g_z g_{\bar{z}})\phi^2\left(\frac{(\phi\sigma)'}{\phi^2}\right)' = 0\,. \tag{29}$$

Where the prime means differentiation with respect to g. The above relations imply that the determinant of the coefficients of $(g_{y\bar{y}} + g_{z\bar{z}})$ and $(g_y g_{\bar{y}} + g_z g_{\bar{z}})$ is zero i.e.

$$\left((1+\sigma^2)\phi^2\right)'\left(\frac{(\phi\sigma)'}{\phi^2}\right)' + 2\left[(1+\sigma^2)\frac{\phi'}{\phi} + \sigma\sigma'\right]'(\phi\sigma)' = 0\,. \tag{30}$$

We shall determine ϕ and σ from the above equation (30), let $(\phi\sigma) = c$, where c is constant, then $(\phi\sigma)' = 0$,

$$\begin{aligned}&\left((1+\sigma^2)\phi^2\right)'\left(\frac{(\phi\sigma)'}{\phi^2}\right)' = 0\,,\\ &\left[(1+\sigma^2)\frac{\phi'}{\phi} + \sigma\sigma'\right]'(\phi\sigma)'.\end{aligned} \tag{31}$$

We suppose

$$\phi = \sqrt{c}e^{-g}\,,\quad \sigma = \sqrt{c}e^{g}\,,\quad then\ \ \rho = \sqrt{c}e^{g+ia}\,. \tag{32}$$

Applying theorem (1) to ϕ and ρ of equation (32), then we get

$$\phi^I = \frac{\sqrt{c}e^{-g}}{1+ce^{2g}}\,,\quad \rho^I = \frac{\sqrt{c}e^{g-ia}}{1+ce^{2g}}\,,\qquad \bar{\rho}^I = \frac{\sqrt{c}e^{g+ia}}{1+ce^{2g}}\,. \tag{33}$$

Equations (32) and (33) is a new class of solutions of Yang-Mills equations for self-dual $SU(2)$ gauge fields.

4. Exact solutions for self-dual $SU(2)$ gauge fields on Euclidean space when ρ is a complex analytic function

We reduce the equations for self-dual $SU(2)$ gauge fields on Euclidean space to the following equations

$$\begin{aligned}&\frac{1}{2}(1+\rho\bar{\rho})\partial_\mu\partial_\mu ln\phi + \frac{1}{2}\rho\partial_\mu\partial_\mu\bar{\rho} + \frac{\bar{\rho}}{\emptyset}\left[\phi_y\rho_{\bar{y}} + \phi_z\rho_{\bar{z}}\right] + \frac{\rho}{\phi}\left[\phi_y\bar{\rho}_{\bar{y}} + \phi_z\bar{\rho}_{\bar{z}}\right] +\\ &\qquad\qquad + \left[\bar{\rho}_y\rho_{\bar{y}} + \bar{\rho}_z\rho_{\bar{z}}\right] = 0,\\ &\frac{\phi}{2}\partial_\mu\partial_\mu\rho + \frac{\rho}{2}\partial_\mu\partial_\mu\emptyset - \frac{2\rho}{\phi}\left(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}\right) + \left(\phi_{\bar{y}}\rho_y + \phi_{\bar{z}}\rho_z - \phi_y\rho_{\bar{y}} - \phi_z\rho_{\bar{z}}\right) = 0.\end{aligned} \tag{34}$$

When ρ is a complex analytic function of y and z, then we have

$$\rho_{\bar{y}} = \rho_{\bar{z}} = 0\,,\qquad \rho_{y\bar{y}} + \rho_{z\bar{z}} = 0\,. \tag{35}$$

Then, the self-dual Yang-Mills equations (34) takes the form

$$\phi(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - (\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) = 0, \tag{36}$$

$$\rho(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - \frac{2\rho}{\phi}(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) + (\rho_y\phi_{\bar{y}} + \rho_z\phi_{\bar{z}}) = 0\,. \tag{37}$$

We consider now two cases:

(a) Let $\rho = \rho(\phi)$, then we find

$$\rho_y = \rho'\phi_y\,,\qquad \rho_z = \rho'\phi_z\,. \tag{38}$$

Then the two equations (36) and (37) becomes

$$\phi(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - (\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) = 0, \tag{39}$$

$$\rho(\phi_{y\bar{y}} + \phi_{z\bar{z}}) + (\rho' - \frac{2\rho}{\phi})(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) = 0. \tag{40}$$

If we do not consider the case $(\phi_{y\bar{y}} + \phi_{z\bar{z}}) = 0$ and $(\emptyset_y \phi_{\bar{y}} + \emptyset_z \phi_{\bar{z}}) = 0$, then we have

$$\phi\rho' - \rho = 0\,, \tag{41}$$

by integration we obtain

$$\rho = c\phi\,, \quad \text{where c is complex constant.} \tag{42}$$

Both equations (39) and (40) reduce to the same equation. A solution is given by

$$\phi_y = \phi_z\,, \qquad \phi_{\bar{y}} = -\phi_{\bar{z}}\;. \tag{43}$$

The solution class is given by [26]

$$\phi = F(\mathcal{Y} + \mathcal{Z}, \bar{\mathcal{Y}} - \bar{\mathcal{Z}})\,, \tag{44}$$

where F is an arbitrary function, equations (42) and (44) gives a new class of solutions of Yang-Mills equations for self-dual $SU(2)$ gauge fields. Applying theorem (1) to ϕ and ρ of equations (42) and (44) then we get

$$\phi^I = \frac{F}{1+c\bar{c}F^2}\,, \quad \rho^I = \frac{\bar{c}F}{1+c\bar{c}F^2}\,, \qquad \bar{\rho}^I = \frac{cF}{1+c\bar{c}F^2}\;. \tag{45}$$

(b) Let us make the ansatz

$$\phi = \phi(g), \qquad \rho = e^{ia}\sigma(g). \tag{46}$$

Where $g = g(x_1, x_2, x_3, x_4)$ is a real function of x_μ, $\mu = 1,2,3,4$, ϕ and σ are real functions of g, and a is a real constant. Then equation (36) and (37) gives the relations

$$\phi\phi'(g_{y\bar{y}} + g_{z\bar{z}}) + (g_y g_{\bar{y}} + g_z g_{\bar{z}})[\phi\phi'' - \phi'^2] = 0\,, \tag{47}$$

$$\sigma\phi'(g_{y\bar{y}} + g_{z\bar{z}}) + (g_y g_{\bar{y}} + g_z g_{\bar{z}})\left(\sigma\phi'' - \frac{2\sigma\phi'^2}{\phi} + \phi'\sigma'\right) = 0\,. \tag{48}$$

Where the prime means differentiation with respect to g. The above relations imply that the determinant of the coefficients of $(g_{y\bar{y}} + g_{z\bar{z}})$ and $(g_y g_{\bar{y}} + g_z g_{\bar{z}})$ is zero i.e.

$$\frac{\sigma'}{\sigma} = \frac{\phi'}{\phi}\,, \tag{49}$$

by integration (49) we obtain

$$\sigma(g) = c\phi(g), \qquad \rho = ce^{ia}\phi(g). \tag{50}$$

Applying theorem (1) to ϕ and ρ of equation (50), then we get

$$\phi^I = \frac{\phi(g)}{1+c^2\phi^2(g)}\,, \quad \rho^I = \frac{ce^{-ia}\phi(g)}{1+c^2\phi^2(g)}\,, \qquad \bar{\rho}^I = \frac{ce^{ia}\phi(g)}{1+c^2\phi^2(g)}\;. \tag{51}$$

Equations (50) and (51) is a new class of solutions of Yang-Mills equations for self-dual $SU(2)$ gauge fields.

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